Necessary and sufficient conditions for the existence of the q-optimal measure

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Abstract

This paper presents the general form and essential properties of the q-optimal measure following the approach of Delbaen & Schachermayer (1996) and proves its existence under mild conditions. Most importantly, it states a necessary and sufficient condition for a candidate measure to be the q-optimal measure in the case even of signed measures. Finally, an updated characterization of the q-optimal measure for continuous asset price processes is presented in the light of the counterexample appearing in Cerny & Kallsen (2006) concerning Hobson's (2004) approach.

Keywords: q-optimal martingale measure, uniformly integrable martingale, signed local martingale measures, incomplete markets.

1 Introduction

In an incomplete market, the choice of the equivalent martingale measure (EMM) for the underlying price process is not unique. Over the last twenty years, many authors have proposed different preference based criteria in order to choose a 'suitable' pricing measure from the class of EMMs. Two of the most popular choices are the minimal entropy EMM, see for example Frittelli (2000), and the variance optimal EMM, see Delbaen & Schachermayer (1996) and Schweizer (1996).

Recently, Hobson (2004) proposed a characterisation of the q-optimal measure, for a wide range of choices of EMMs, which includes the two aforementioned measures. The notion of q-optimality is linked to the unique EMM with minimal q-moment (if $q > 1$) or minimal relative entropy (if $q = 1$). Hobson's (2004) approach to identifying the q-optimal measure (through a so-called fundamental equation) suggests a relaxation of an essential condition appearing in Delbaen & Schachermayer (1996). This condition states that for the case $q = 2$, the Radon-Nikodym process, whose last element is the density of the candidate measure, is a uniformly integrable martingale with respect to any EMM with a bounded

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second moment. Hobson (2004) alleges that it suffices to show that the above is true only with respect to the candidate measure itself and extrapolates for the case $q > 1$. Cerny & Kallsen (2006) however presented a counterexample (for $q = 2$) which demonstrates that the above relaxation does not hold in general. The case $q = 1$ is covered by Grandits & Rheinländer (2002).

This paper follows the approach of Delbaen & Schachermayer (1996) to describe and present the essential properties of the q-optimal measure (with $q > 1$) by extending the definition to include also signed local martingale measures, see for example Grandits & Rheinländer (2002). In the light of the counterexample appearing in Cerny & Kallsen (2006), the analogous sufficient condition for $q > 1$ is presented to guarantee that a candidate measure is indeed the q-optimal measure. Most importantly, it is proven here that the condition under consideration is also necessary for the identification of the q-optimal measure. Furthermore, the information concerning the form of the q-optimal measure helps us identify the constant appearing in the so-called fundamental representation equation, see Hobson (2004), which determines when a candidate measure has the q-optimality property and an updated characterization of the q-optimal measure is given.

2 Main Result

Let us consider an $\mathbb{R}^d$-valued, locally bounded, cadlag semimartingale $S := \{S_t\}_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. It is assumed that S models the evolution of d discounted stock price processes. Furthermore, let us consider K_0, a linear subspace of $L^\infty(\mathbb{P})$, which is spanned by simple stochastic integrals of the form (dot product)

$$h = \phi(S_{\tau_2} - S_{\tau_1})$$

where τ_1 and τ_2 are stopping times such that: (i) $\tau_1 \leq \tau_2$ a.s., (ii) the stopped process $S^{\tau_2} := \{S_{\tau_2 \wedge t}\}_{t\geq 0}$ is bounded. Moreover, ϕ is assumed to be a bounded $\mathbb{R}^d$-valued $\mathcal{F}_{\tau_1}$-measurable function. Then, we remind ourselves of the following well-known definitions:

Definition 2.1 *A probability measure $\mathbb{Q}$ on $\mathcal{F}$ with density $u := \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^1(\mathbb{P})$ is a local martingale measure for S iff $\mathbb{Q}$ vanishes on K_0 i.e.,* $\mathbb{E}[uh] = 0$ for all $h \in K_0$.

Definition 2.2 *The following collection of random variables*

$$\mathcal{M}^s(\mathbb{P}) = \{u \in L^1(\mathbb{P}) : \mathbb{E}[uh] = 0 \textit{ for any } h \in K_0, \textit{ and } \mathbb{E}[u] = 1\}$$

is called the set of signed local martingale measures for the process S.

Moreover, the set of absolutely continuous (resp. equivalent) local martingale measures $\mathcal{M}(\mathbb{P})$ (resp. $\mathcal{M}^e(\mathbb{P})$) for the process S is defined as the intersection of $\mathcal{M}^s(\mathbb{P})$ with the positive (resp. strictly positive) orthant of $L^1(\mathbb{P})$. Recall also here that $\mathcal{M}^s(\mathbb{P}) \cap L^q(\mathbb{P})$ is closed in $L^q(\mathbb{P})$ and that it has a unique element of minimal $L^q(\mathbb{P})$-norm (provided that $\mathcal{M}^s(\mathbb{P}) \cap L^q(\mathbb{P}) \neq \emptyset$) due to the strict convexity of the norm.

Definition 2.3 *Suppose that $\mathcal{M}^s(\mathbb{P}) \cap L^q(\mathbb{P}) \neq \emptyset$ and $q > 1$. Then, the unique element of $\mathcal{M}^s(\mathbb{P})$ with minimal $L^q(\mathbb{P})$-norm is called the q-optimal signed local martingale measure for the process S.*

One can then identify the general form of the q-optimal measure following the approach of Delbaen & Schachermayer (1996). Although this result is known in the literature, see for example Grandits (1999), it is important in the author's view to present a relevant proof here so as to be able to proceed with the construction of the necessary and sufficient condition for the existence of the q-optimal measure in the general framework of signed measures.

It is noted though that for $q \neq 2$, one operates in Banach spaces instead of Hilbert spaces since the dual of $L^q(\mathbb{P})$ is $L^p(\mathbb{P})$, where $p = \frac{q}{q-1}$. Nevertheless, it is possible to extend Delbaen & Schachermayer (1996) results with a careful approach. Let $\bar{K}_0$ denote the closure of K_0 in $L^p(\mathbb{P})$ and $\bar{K}$ denote the closure of the span of K_0 and the constants also in $L^p(\mathbb{P})$. Then, the annihilator of $\bar{K}_0$, which is denoted by $\bar{K}_0^\alpha$, is in $L^q(\mathbb{P})$. Let also $||\cdot||_p$ and $||\cdot||_q$ denote the $L^p(\mathbb{P})$-norm and $L^q(\mathbb{P})$-norm respectively.

Theorem 2.4 *Fix $q > 1$. The following statements hold:*

(a) $\mathcal{M}^s(\mathbb{P}) \cap L^q(\mathbb{P}) \neq \emptyset$ iff $\bar{K}_0$ does not contain the constant function 1.

(b) If $\mathcal{M}^s(\mathbb{P}) \cap L^q(\mathbb{P}) \neq \emptyset$, then the probability measure $\mathbb{Q}^*$ defined by

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} := \frac{g^*}{\mathbb{E}[g^*]},$$

where $g^* := \mathrm{sgn}(1-f)|1-f|^{p-1}$ and f is the unique element of $\bar{K}_0$ with the property

$$||1-f||_p = \inf_{h \in \bar{K}_0} ||1-h||_p,$$

is the unique element of $\bar{K}_0^\alpha$ with minimal $L^q(\mathbb{P})$-norm.

Proof (a) The linear functional $\varphi \in \bar{K}_0^\alpha$ with $\varphi(1) = 1$ is well defined and continuous on $\bar{K}$ iff $1 \notin \bar{K}_0$.

(b) Let f be the unique element of $\bar{K}_0$ such that $||1-f||_p = \inf_{h \in \bar{K}_0} ||1-h||_p$ (uniqueness is due to the strict convexity of the $L^p(\mathbb{P})$-norm). Let $g := 1 - f$, and observe that for any other $h \in \bar{K}_0$ and $t \in \mathbb{R}$

$$||g + th||_p^p \geq ||g||_p^p$$

holds. As a result, we obtain

$$\frac{d}{dt}||g + th||_p^p|_{t=0} = 0 \qquad \Rightarrow \qquad p\mathbb{E}[\mathrm{sgn}(g)|g|^{p-1}h] = 0.$$

Set $g^* = \mathrm{sgn}(g)|g|^{p-1}$ and observe that $\mathbb{E}[g^*] = \mathbb{E}[g^*(1-f)] = ||g||_p^p > 0$. Thus, $\frac{g^*}{\mathbb{E}[g^*]} \in \bar{K}_0^\alpha$ and $\mathbb{E}[\frac{g^*}{\mathbb{E}[g^*]}] = 1$. Furthermore, we calculate

$$||\frac{g^*}{\mathbb{E}[g^*]}||_q^q = \frac{1}{||g||_p^{pq}}\mathbb{E}[|g|^{q(p-1)}] = \frac{1}{||g||_p^q} < \infty.$$

which implies that $\mathbb{Q}^* \in \mathcal{M}^s(\mathbb{P}) \cap L^q(\mathbb{P})$. Finally, for any element $u \in \bar{K}_0^\alpha$ with $\mathbb{E}[u] = 1$ (i.e., any signed local martingale measure with density in $L^q(\mathbb{P})$) we obtain

$$\mathbb{E}[ug] = \mathbb{E}[u(1-f)] = 1$$

and thus Hölder inequality yields

$$1 \leq ||u||_q ||g||_p \quad \Rightarrow \quad ||u||_q \geq \frac{1}{||g||_p} = ||\frac{g^*}{\mathbb{E}[g^*]}||_q.$$

and that concludes the proof. ∎

It is the general form of the q-optimal measure presented in Theorem 2.4 that holds the key to obtaining the necessary and sufficient condition for proving the q-optimality property of a candidate measure. It is therefore important to recall here the counterexample from Cerny & Kallsen (2006). The counterexample shows that (for $q = 2$) a candidate measure may not be the q-optimal measure if we only prove that the Radon-Nikodym process, whose last element is the density of the candidate measure, is a uniformly integrable martingale with respect to the candidate measure itself. Therefore, we still require the condition set by Delbaen & Schachermayer (1996), i.e. the corresponding Radon-Nikodym process should be a uniformly integrable martingale with respect to any EMM with a bounded second moment. The main Theorem of this section follows.

Theorem 2.5 *Let $q > 1$ and suppose that there exists $\mathbb{Q}^* \in \mathcal{M}^s(\mathbb{P}) \cap L^q(\mathbb{P})$ defined by*

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} := \frac{g^*}{\mathbb{E}[g^*]}$$

The following statements hold:

(i) if $\mathbb{Q}^*$ is the q-optimal measure, then $\mathbb{E}_\mathbb{Q}[sgn(g^*)|g^*|^{q-1}] = 1$ for every $\mathbb{Q} \in \mathcal{M}^s(\mathbb{P}) \cap L^q(\mathbb{P})$;

(ii) conversely, if $\mathbb{E}_\mathbb{Q}[sgn(g^*)|g^*|^{q-1}] = 1$ for every $\mathbb{Q} \in \mathcal{M}^s(\mathbb{P}) \cap L^q(\mathbb{P})$, then $\mathbb{Q}^*$ is the q-optimal martingale measure.

Proof (i) If $\mathbb{Q}^*$ is the q-optimal measure, then Theorem 2.4 asserts that

$$g^* = \mathrm{sgn}(1-f)|1-f|^{p-1},$$

where $||1-f||_p = \inf_{h \in \bar{K}_0} ||1-h||_p$, and thus

$$\mathbb{E}[u\, sgn(g^*)|g^*|^{q-1}] = \mathbb{E}[u\, sgn(1-f)|1-f|^{(p-1)(q-1)}] = \mathbb{E}[u(1-f)] = 1,$$

for any $u \in \bar{K}_0^\alpha$ with $\mathbb{E}[u] = 1$.

(ii) If $\mathbb{E}_{\mathbb{Q}}[sgn(g^*)|g^*|^{q-1}] = 1$ for every $\mathbb{Q} \in \mathcal{M}^s(\mathbb{P}) \cap L^q(\mathbb{P})$, then

$$\mathbb{E}[\frac{g^*}{\mathbb{E}[g^*]} sgn(g^*)|g^*|^{q-1}] = 1 \quad \Longrightarrow \mathbb{E}[g^*] = \mathbb{E}[|g^*|^q] > 0.$$

Set $\mu := \mathbb{E}[g^*] = \mathbb{E}[|g^*|^q]$ and observe that

$$\mathbb{E}[|\frac{d\mathbb{Q}^*}{d\mathbb{P}}|^q] = \mathbb{E}[|\frac{g^*}{\mathbb{E}[g^*]}|^q] = \frac{\mu}{\mu^q} = \mu^{1-q} = \mu^{-q/p}.$$

Moreover, for any $\mathbb{Q} \in \mathcal{M}^s(\mathbb{P}) \cap L^q(\mathbb{P})$,

$$1 = \mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}} sgn(g^*)|g^*|^{q-1}] \leq ||\frac{d\mathbb{Q}}{d\mathbb{P}}||_q|| sgn(g^*)|g^*|^{q-1}||_p = ||\frac{d\mathbb{Q}}{d\mathbb{P}}||_q(\mathbb{E}[|g^*|^q])^{1/p}$$

and thus

$$||\frac{d\mathbb{Q}}{d\mathbb{P}}||_q \geq \mu^{-1/p} \qquad \Longrightarrow \qquad \mathbb{E}[|\frac{d\mathbb{Q}}{d\mathbb{P}}|^q] \geq \mu^{-q/p} = \mathbb{E}[|\frac{d\mathbb{Q}^*}{d\mathbb{P}}|^q]$$

and that concludes the proof. ∎

Remark 2.6 The condition $\mathbb{E}_{\mathbb{Q}}[sgn(g^*)|g^*|^{q-1}] = 1$, which is translated as $\mathbb{E}_{\mathbb{Q}}[(g^*)^{q-1}] = 1$ for the EMMs case, implies that the stochastic process $(\hat{V}^{\text{opt}})^{q-1} = \{\mathbb{E}_{\mathbb{Q}}[(V_\infty^{\text{opt}})^{q-1}|\mathcal{F}_t]\}_{0 \leq t \leq \infty}$ is a uniformly integrable $\mathbb{Q}$-martingale with respect to any $\mathbb{Q} \in \mathcal{M}^e(\mathbb{P}) \cap L^q(\mathbb{P})$, see Lemma 2.12. Moreover, for $q = 2$, one obtains that the corresponding Radon-Nikodym process, whose last element is the density $\frac{d\mathbb{Q}^*}{d\mathbb{P}}$, is a uniformly integrable $\mathbb{Q}$-martingale with respect to any $\mathbb{Q} \in \mathcal{M}^e(\mathbb{P}) \cap L^2(\mathbb{P})$ and this is a necessary and sufficient condition for $\mathbb{Q}^*$ to be the q-optimal (local) martingale measure.

Let us turn our attention now to the case where S is a continuous adapted stochastic process. Then, one can prove that $\mathbb{Q}^*$ is a probability measure equivalent to $\mathbb{P}$. This result is also known in the literature, see for example Grandits & Rheinländer (2002), but it is presented here as the generalisation of Delbaen & Schachermayer (1996) technique for completeness of this section.

Note also that the notation $(\varphi{\cdot}S)_t \in \bar{K}_0$ is used as a shorthand notation for the stochastic integral

$$(\varphi \cdot S)_t = \int_0^t \varphi_u dS_u$$

for every $0 \leq t \leq \infty$, where the process $\varphi \in \mathcal{H}_p$, i.e. it satisfies

$$\mathbb{E}[(\int_0^\infty \varphi_t^2 d[S]_t)^{p/2}] < \infty.$$

Theorem 2.7 *Fix $q > 1$. Let us assume that S is a continuous process and that $\mathcal{M}^s(\mathbb{P}) \cap L^q(\mathbb{P}) \neq \emptyset$. Then, the q-optimal signed local martingale measure $\mathbb{Q}^*$ is a well-defined probability measure absolutely continuous with $\mathbb{P}$.*

Proof In order to show that $\frac{d\mathbb{Q}^*}{d\mathbb{P}}$ is non-negative, it suffices to prove that $f \leq 1$ (a.s.).

Let us assume (on the contrary) that there exists $\epsilon \in (0,1)$ such that $\mathbb{P}(f > 1+\epsilon) > \epsilon$. Then, there exists a simple integrand ϕ such that

(a) $(\phi \cdot S)_\infty \in \bar{K}_0$,

(b) $||(\phi \cdot S)_\infty - f||_p \leq c$, where $c < (\frac{\epsilon}{2})^{p+1}(\sum_{i=1}^{\lceil p/2 \rceil} \binom{p}{2i-1})^{-1}$ and,

(c) $||1 - (\phi \cdot S)_\infty||_p \leq 1$ (since $||1-f||_p \leq 1 < 1+c$).

Then, we observe that

$$\mathbb{P}((\phi \cdot S)_\infty > 1 + \frac{\epsilon}{2}) \geq \mathbb{P}(f > 1+\epsilon) - \mathbb{P}(|f - (\phi \cdot S)_\infty| > \frac{\epsilon}{2})$$

and since

$$\mathbb{P}(|f - (\phi \cdot S)_\infty| > \frac{\epsilon}{2}) \leq (\frac{2}{\epsilon})^p \mathbb{E}[|f - (\phi \cdot S)_\infty|^p] \leq (\frac{2}{\epsilon})^p c^p$$

we conclude that

$$\mathbb{P}((\phi \cdot S)_\infty > 1 + \frac{\epsilon}{2}) \geq \epsilon - (\frac{2}{\epsilon})^p (\frac{\epsilon}{2})^{(p+1)p} (\sum_{i=1}^{\lceil p/2 \rceil} \binom{p}{2i-1})^{-p} \geq \frac{\epsilon}{2}.$$

Moreover, we define the stopping time $\tau = \inf\{t \geq 0 : (\phi \cdot S)_t > 1\}$. Then,

$$|1-(\phi \cdot S)_\infty|^p = |1-(\phi \cdot S)_\tau|^p \mathbb{1}_{\{\tau=\infty\}} + |1-(\phi \cdot S)_\infty|^p \mathbb{1}_{\{\tau<\infty\}} = |1-(\phi \cdot S)_\tau|^p + |1-(\phi \cdot S)_\infty|^p \mathbb{1}_{\{\tau<\infty\}}$$

since for $\tau < \infty$ we have $1 - (\phi \cdot S)_\tau = 0$ due to the continuity of S. Hence,

$$\begin{aligned}
||1 - (\phi \cdot S)_\infty||_p^p &= ||1 - (\phi \cdot S)_\tau||_p^p + \mathbb{E}[|1 - (\phi \cdot S)_\infty|^p \mathbb{1}_{\{\tau<\infty\}}] \\
&\geq ||1 - (\phi \cdot S)_\tau||_p^p + \mathbb{E}[|1 - (\phi \cdot S)_\infty|^p \mathbb{1}_{\{(\phi \cdot S)_\infty > 1 + \frac{\epsilon}{2}\}}] \\
&\geq ||1 - (\phi \cdot S)_\tau||_p^p + (\frac{\epsilon}{2})^p \mathbb{P}((\phi \cdot S)_\infty > 1 + \frac{\epsilon}{2}) \\
&\geq ||1 - (\phi \cdot S)_\tau||_p^p + (\frac{\epsilon}{2})^{p+1}.
\end{aligned}$$

Note also that due to Minkowski inequality

$$||1 - (\phi \cdot S)_\infty||_p \leq ||1 - f||_p + ||(\phi \cdot S)_\infty - f||_p \leq ||1 - f||_p + c$$

which implies

$$
\begin{aligned}
||1-f||_p^p &\geq ||1-(\phi\cdot S)_\infty||_p^p + \sum_{i=1}^{p}\binom{p}{i}(-1)^i||1-(\phi\cdot S)_\infty||_p^{p-i}c^i \\
&\geq ||1-(\phi\cdot S)_\infty||_p^p - \sum_{i=1}^{\lceil p/2\rceil}\binom{p}{2i-1}c^{2i-1} \\
&\geq ||1-(\phi\cdot S)_\tau||_p^p + (\frac{\epsilon}{2})^{p+1} - \sum_{i=1}^{\lceil p/2\rceil}\binom{p}{2i-1}c^{2i-1} \\
&\geq ||1-(\phi\cdot S)_\tau||_p^p + (\frac{\epsilon}{2})^{p+1} - c\sum_{i=1}^{\lceil p/2\rceil}\binom{p}{2i-1} \\
&> ||1-(\phi\cdot S)_\tau||_p^p
\end{aligned}
$$

which is a contradiction since f is the unique element of $\bar{K}_0$ with the property $||1-f||_p = \inf_{h\in\bar{K}_0}||1-h||_p$. ∎

Theorem 2.4 states that $f \in \bar{K}_0$, therefore under the assumption that S is a semi-martingale, we can represent

$$g = 1 - f = 1 - (\psi\cdot S)_\infty.$$

Moreover, we fix $\mathbb{Q} \in \mathcal{M}^e(\mathbb{P}) \cap L^q(\mathbb{P})$ and for every $t \geq 0$ we define

$$
\begin{aligned}
V_\infty^{opt} := \frac{g^*}{\mathbb{E}[g^*]} = \frac{g^{p-1}}{\mathbb{E}[g^*]} \quad &\& \quad V_t^{opt} = \mathbb{E}[V_\infty^{opt}|\mathcal{F}_t], \\
X_t := \mathbb{E}[g^{p-1}|\mathcal{F}_t] = \mathbb{E}[g^*]V_t^{opt} \quad &\& \quad Y_t = 1-(\psi\cdot S)_t = \mathbb{E}_{\mathbb{Q}}[g|\mathcal{F}_t]
\end{aligned}
$$

and the stopping times

$$\tau := \inf\{t\geq 0 : X_t = 0\} \quad \& \quad \sigma := \inf\{t \geq 0 : Y_t = 0\}.$$

Note that the processes X and Y are non-negative supermartingales with non-negative last elements X_∞ and Y_∞, therefore when any of their paths hits zero, it stays at zero. Furthermore, the continuity of Y implies that the stopping time σ is predictable. As a result, the following lemmas (2.8 and 2.10) can be proved in a similar fashion as in Delbaen & Schachermayer (1996).

Lemma 2.8 *Fix $q > 1$. Let us assume that S is a continuous semi-martingale and that $\mathcal{M}^e(\mathbb{P}) \cap L^q(\mathbb{P}) \neq \emptyset$. Then, $\tau = \sigma$.*

Proof Our aim is to prove that $\mathbb{P}(\sigma < \tau) = \mathbb{P}(\sigma > \tau) = 0$. Consider the set $\{\sigma < \tau\}$, then

$$0 < X_\sigma = \mathbb{E}[X_\infty|\mathcal{F}_\sigma] = \mathbb{E}[g^{p-1}|\mathcal{F}_\sigma] = \mathbb{E}[(1-(\psi\cdot S)_\sigma)^{p-1}|\mathcal{F}_\sigma]$$

since $1-(\psi\cdot S)_\infty = 1-(\psi\cdot S)_\sigma$ on $\{\sigma<\tau\}\subset\{\sigma<\infty\}$ and thus

$$0 < X_\sigma = (1-(\psi\cdot S)_\sigma)^{p-1} = 0 \qquad \text{(contradiction)}$$

which implies $\mathbb{P}(\sigma<\tau)=0$. Now consider the set $\{\sigma>\tau\}\subset\{\tau<\infty\}$ and observe that

$$0 = X_\tau = \mathbb{E}[X_\infty|\mathcal{F}_\tau] = \mathbb{E}[g^{p-1}|\mathcal{F}_\tau]$$

which implies $g=0$ on $\{\sigma>\tau\}$. Thus, since $Y_\tau = \mathbb{E}_{\mathbb{Q}}[g|\mathcal{F}_\tau]$, we obtain $Y_\tau=0$ on $\{\sigma>\tau\}$ (contradiction) which implies $\mathbb{P}(\sigma>\tau)=0$. ∎

Corollary 2.9 *The martingale V^{opt} is continuous at time $t=\tau$ and the stopping time τ is predictable and thus is announced by the sequence $\tau_n = \inf\{t\geq 0 : V_t^{opt}\leq \frac{1}{n}\}\wedge n$.*

Lemma 2.10 *Let $M := \{M_t\}_{0\leq t\leq\infty}$ be a qth integrable martingale such that $M_0>0$. Let also $\tau = \inf\{t\geq 0 : M_t = 0\}$ be a predictable stopping time announced by a sequence of stopping times $\{\tau_n\}_{n\geq 1}$. Then,*

$$\mathbb{E}[\frac{M_\infty^q}{M_{\tau_n}^q}|\mathcal{F}_{\tau_n}] \to \infty$$

on the set $\{M_\tau = 0\}$.

Proof First observe that

$$\mathbb{1} = \mathbb{E}[\frac{M_\infty}{M_{\tau_n}}|\mathcal{F}_{\tau_n}] = \mathbb{E}[\frac{M_\infty}{M_{\tau_n}}\mathbb{1}_{\{M_\tau\neq 0\}}|\mathcal{F}_{\tau_n}] \leq \mathbb{E}[(\frac{M_\infty}{M_{\tau_n}})^q|\mathcal{F}_{\tau_n}]^{1/q}\mathbb{E}[\mathbb{1}_{\{M_\tau\neq 0\}}|\mathcal{F}_{\tau_n}]^{1/p}$$

and then recall that $\mathbb{E}[\mathbb{1}_{\{M_\tau\neq 0\}}|\mathcal{F}_{\tau_n}]$ tends to zero on $\{M_\tau=0\}$. ∎

Theorem 2.11 *Fix $q>1$. Let us assume that S is a continuous semi-martingale and that $\mathcal{M}^e(\mathbb{P})\cap L^q(\mathbb{P})\neq\emptyset$. Then, the q-optimal local martingale measure $\mathbb{Q}^*$ is in fact equivalent to $\mathbb{P}$.*

Proof Let us assume on the contrary that $\mathbb{P}[X_\tau=0]>0$ and observe that for the uniformly integrable martingale V, where $V_t := \mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t]$ for all $0\leq t\leq\infty$ and $\mathbb{Q}\in\mathcal{M}^e(\mathbb{P})\cap L^q(\mathbb{P})$, we have $\inf_{t\geq 0}V_t>0$ and $\sup_{t\geq 0}\mathbb{E}[(V_\infty)^q|\mathcal{F}_t]<\infty$ (both inequalities hold a.s.). In view of Lemma 2.10, one expects that for a large enough n the set

$$A = \left\{\sup_{t\geq 0}\frac{\mathbb{E}[(V_\infty)^q|\mathcal{F}_t]}{(V_t)^q} < \frac{\mathbb{E}[(V_\infty^{\text{opt}})^q|\mathcal{F}_{\tau_n}]}{(V_{\tau_n}^{\text{opt}})^q}\right\}$$

is non empty, thus

$$A_n = \left\{\frac{\mathbb{E}[(V_\infty)^q|\mathcal{F}_{\tau_n}]}{(V_{\tau_n})^q} < \frac{\mathbb{E}[(V_\infty^{\text{opt}})^q|\mathcal{F}_{\tau_n}]}{(V_{\tau_n}^{\text{opt}})^q}\right\}$$

is non empty in $\mathcal{F}_{\tau_n}$. Then, the martingale

$$\bar{V}_t = \begin{cases} V_t^{\text{opt}}, & t<\tau_n, \\ \frac{V_t V_{\tau_n}^{\text{opt}}}{V_{\tau_n}}, & \text{for } t\geq\tau_n \text{ on the set } A_n, \\ V_t^{\text{opt}}, & \text{for } t\geq\tau_n \text{ on the complement of the set } A_n, \end{cases}$$

defines an equivalent martingale measure $\bar{\mathbb{Q}}$ to $\mathbb{P}$ such that $||\bar{V}_\infty||_q < ||V_\infty^{\text{opt}}||_q$ which is clearly a contradiction. ∎

The last Lemma of this section provides the connection between the condition appearing in Theorem 2.5 and the behaviour of $\{(\hat{V}_t^{\text{opt}})^{q-1}\}_{0\le t\le\infty}$ as defined below.

Lemma 2.12 *Fix $q > 1$. Let $\mathcal{M}^e(\mathbb{P}) \cap L^q(\mathbb{P}) \neq \emptyset$ and fix $\hat{\mathbb{Q}} \in \mathcal{M}^e(\mathbb{P}) \cap L^q(\mathbb{P})$. Let us define the process $\hat{V}$ by $\hat{V}_t^{\text{opt}} := (\mathbb{E}_{\hat{\mathbb{Q}}}[(V_\infty^{\text{opt}})^{q-1}|\mathcal{F}_t])^{1/(q-1)}$ for every $t \geq 0$. Then,*

$$(\hat{V}_t^{\text{opt}})^{q-1} = ||V_\infty^{\text{opt}}||_q^q + (\varphi \cdot S)_t \tag{2.1}$$

where the stochastic integral $(\varphi \cdot S)$ is well defined, i.e. $\varphi \in \mathcal{H}_p$, and is a uniformly integrable $\mathbb{Q}$-martingale for every $\mathbb{Q} \in \mathcal{M}^e(\mathbb{P}) \cap L^q(\mathbb{P})$. Furthermore, the choice of φ is independent of the choice of $\hat{\mathbb{Q}} \in \mathcal{M}^e(\mathbb{P}) \cap L^q(\mathbb{P})$.

Proof Recall that $g \in \bar{K}$ and $(g^*)^{q-1} = g$ which imply that there exists a sequence $\{g_i\}_{i\geq 1} \in K$ that converges to $(V_t^{\text{opt}})^{q-1}$ in $L^p(\mathbb{P})$. Moreover, we observe that

$$\mathbb{E}_{\hat{\mathbb{Q}}}[g_i - (V_\infty^{\text{opt}})^{q-1}] = \mathbb{E}[(g_i - (V_\infty^{\text{opt}})^{q-1})\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}] \leq ||g_i - (V_\infty^{\text{opt}})^{q-1}||_p||\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}||_q$$

which implies convergence in $L^1(\hat{\mathbb{Q}})$. Note that if we choose to represent each $g_i \in K$ as follows

$$g_i = \delta_i + (\phi_i \cdot S)$$

where δ_i denotes the real number in the representation, we obtain as a result that

$$\begin{aligned}\lim_{i\to\infty} \delta_i = \lim_{i\to\infty} \mathbb{E}_{\hat{\mathbb{Q}}}[g_i] = \mathbb{E}_{\hat{\mathbb{Q}}}[(V_\infty^{\text{opt}})^{q-1}] = \mathbb{E}[(V_\infty^{\text{opt}})^{q-1}\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}] = \frac{1}{(\mathbb{E}[g^*])^{q-1}}\\ = \mathbb{E}[(V_\infty^{\text{opt}})^{q-1}\frac{d\mathbb{Q}^*}{d\mathbb{P}}] = \frac{1}{||g||_p^q} = ||V_\infty^{\text{opt}}||_q^q,\end{aligned}$$

so the process $\{g_i - \delta_i\}_{1\le i\le\infty}$ converges in $L^1(\hat{\mathbb{Q}})$ to $(V_\infty^{\text{opt}})^{q-1} - ||V_\infty^{\text{opt}}||_q^q$. Thus, following once more the approach of Delbaen & Schachermayer (1996), one obtains that the choice of φ is independent of the choice of $\hat{\mathbb{Q}}$ since the process $(\varphi \cdot S)$ is a uniformly integrable $\mathbb{Q}$-martingale for every $\mathbb{Q} \in \mathcal{M}^e(\mathbb{P}) \cap L^q(\mathbb{P})$ converging to $(V_\infty^{\text{opt}})^{q-1} - ||V_\infty^{\text{opt}}||_q^q$ in $L^1(\mathbb{Q})$. ∎

3 Continuous Univariate Case

Let $T \in (0, \infty]$ denote the termination date of the economy, i.e. we can work under either a finite ($T < \infty$) or an infinite ($T = \infty$) time horizon. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$ be a filtered probability space that satisfies the usual conditions of right-continuity and completeness, where $\mathcal{F} = \mathcal{F}_T$ and $\mathcal{F}_0$ is trivial. Moreover, let $Y := \{Y_t\}_{0\le t\le T}$ denote the volatility of the traded asset S. Suppose that S is a continuous semimartingale governed by the following stochastic differential equation

$$dS_t = \mu(S_t, Y_t, t)dt + \sigma(S_t, Y_t, t)dB_t, \qquad \forall\ t \in [0, T], \tag{3.1}$$

where $B := \{B_t\}_{0\le t\le T}$ is a $\mathbb{P}$-Brownian motion. The semimartingale S admits a Doob-Meyer decomposition given by

$$S = S_0 + A^S + M^S \tag{3.2}$$

where A^S denotes an increasing process and M^S denotes a local martingale. Furthermore, consider the processes

$$\lambda := \frac{\mu}{\sigma}, \qquad \bar{\lambda} := \frac{\lambda}{\sigma} \qquad \& \qquad \bar{\eta} := \frac{\eta}{\sigma}$$

and observe that in the context of equation (3.1)

$$A_t^S := \int_0^t \mu_t dt, \qquad M_t^S = \int_0^t \sigma_t dB_t \qquad \& \qquad A^S = \bar{\lambda}\cdot[M^S]$$

for all $t \in [0, T]$. Then, the following proposition sets out sufficient criteria so that a candidate measure should satisfy in order to be the q-optimal measure.

Proposition 3.1 *Let $T \in (0, \infty]$ and $q > 1$ be fixed. Suppose that there exists a B-integrable, predictable process η such that*

(i) $\mathbb{E}_{\mathbb{Q}}[\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T] = 1$ *for every* $\mathbb{Q} \in \mathcal{M}^e(\mathbb{P}) \cap L^q(\mathbb{P})$,

(ii) $\mathbb{E}[(\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T)^{p-1}]$ *is a non-zero finite constant and,*

(ii) it satisfies

$$\exp(\frac{q}{2}\bar{\lambda}\cdot A_T^S)\mathcal{E}(M^Y)_T = c\mathcal{E}(\bar{\eta}\cdot(M^S + qA^S))_T \exp(-\frac{q-2}{2}\bar{\eta}^2\cdot[M^S]_T), \tag{3.3}$$

where M^Y is a a local martingale with $< M^S, M^Y >= 0$ and c is given by

$$c = 1/\mathbb{E}[(\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T)^{p-1}].$$

Then, $V^{\mathsf{opt}} := \mathcal{E}(-\lambda\cdot B - M^Y)$ is a uniformly integrable $\mathbb{P}$-martingale, and $\mathbb{Q}^*$ with density V_T^{opt} is the q-optimal measure.

Proof The integrability condition imposed on η guarantees the existence of the stochastic integrals appearing in equation (3.3). Then, we calculate

$$\begin{aligned}
V_T^{\mathrm{opt}} &= \mathcal{E}(-\lambda\cdot B - M^Y)_T = \mathcal{E}(-\bar{\lambda}\cdot M^S)_T \exp(-\frac{q}{2}\bar{\lambda}\cdot A_T^S)\exp(\frac{q}{2}\bar{\lambda}\cdot A_T^S)\mathcal{E}(M^Y)_T \\
&= \exp(-\bar{\lambda}\cdot S_T)\exp(-\frac{q-1}{2}\bar{\lambda}\cdot A_T^S)c\mathcal{E}(\bar{\eta}\cdot(M^S+qA^S))_T\exp(-\frac{q-2}{2}\bar{\eta}^2\cdot[M^S]_T) \\
&= \exp((\bar{\eta}-\bar{\lambda})\cdot S_T)\exp(-\frac{q-1}{2}\bar{\lambda}\cdot A_T^S)c\exp((q-1)\bar{\eta}\bar{\lambda}\cdot[M^S]_T)\exp(-\frac{q-1}{2}\bar{\eta}^2\cdot[M^S]_T) \\
&= c\exp((\bar{\eta}-\bar{\lambda})\cdot S_T - \frac{q-1}{2}(\bar{\eta}-\bar{\lambda})^2\cdot[S]_T) = c(\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T)^{p-1}
\end{aligned}$$

and consequently $\mathbb{Q}^* \in \mathcal{M}^e(\mathbb{P})$ since $\mathbb{E}[V_T^{\text{opt}}] = \mathbb{E}[c(\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T)^{p-1}] = 1$ due to condition (ii). Moreover, $\mathbb{Q}^* \in L^q(\mathbb{P})$ since

$$(\frac{d\mathbb{Q}^*}{d\mathbb{P}})^{q-1} = (V_T^{\text{opt}})^{q-1} = c^{q-1}\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T$$

which yields

$$\mathbb{E}[(\frac{d\mathbb{Q}^*}{d\mathbb{P}})^q] = \mathbb{E}_{\mathbb{Q}^*}[(\frac{d\mathbb{Q}^*}{d\mathbb{P}})^{q-1}] = \mathbb{E}_{\mathbb{Q}^*}[c^{q-1}\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T] = c^{q-1} < \infty.$$

Condition (i) and Theorem 2.5 assert that $\mathbb{Q}^*$ is the q-optimal martingale measure. Furthermore, Theorem 2.4 identifies g as the last element $\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T$ of the uniformly integrable $\mathbb{Q}$-martingale $\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)$. ∎

Remark 3.2 Another byproduct of the q-optimal measure comes from

$$1 = \mathbb{E}_{\mathbb{Q}}[\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T] = \mathbb{E}[c(\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T)^{p-1}\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T],$$

which yields

$$\mathbb{E}[(\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T)^p] = \mathbb{E}[(\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T)^{p-1}].$$

A property that holds also due to $\mathbb{E}[g^p] = \mathbb{E}[g^{p-1}(1-f)] = \mathbb{E}[g^{p-1}]$ according to Theorem 2.4.

Remark 3.3 Equation (3.3) in Proposition 3.1 is a generalisation of the Fundamental Equation (1.2) in Hobson (2004) and Equation (3.2) in Cerny & Kallsen. Moreover, condition *(i)* in Proposition 3.1 is the essential difference with Theorem 3.1, page 543, in Hobson (2004) and addresses the issue related to the counterexample presented by Cerny & Kallsen (2006). Condition *(i)* is replaced by the weaker condition

$$\mathbb{E}_{\mathbb{Q}^{(q)}}[\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T] = 1,$$

where $\mathbb{Q}^{(q)}$ is a candidate measure, in Hobson (2004).

Remark 3.4 In Hobson (2004), Y is assumed to be driven by

$$dY_t = \alpha(Y_t,t)dt + \beta(Y_t,t)dW_t, \qquad \forall\, t \in [0,T], \tag{3.4}$$

which implies that $M^Y = \xi \cdot W$, where $W := \{W_t\}_{0\le t\le T}$ is a $\mathbb{P}$-Brownian motions such that $dW_t = \rho_t dB_t + \sqrt{1-\rho_t^2}dZ_t$, B and $Z := \{Z_t\}_{0\le t\le T}$ are independent $\mathbb{P}$-Brownian motions and ρ_t is the instantaneous correlation. It is possible then to identify the constant c_H appearing in Hobson's so-called fundamental representation equation, i.e. equation (1.2), page 538,

$$c_H = \ln c = -\ln(\mathbb{E}[(\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T)^{p-1}]) = -\ln(\mathbb{E}[(\mathcal{E}((q-1)(\bar{\eta}-\bar{\lambda})\cdot S)_T)^p]) \tag{3.5}$$

and observe that indeed

$$\mathbb{E}[(\frac{d\mathbb{Q}^*}{d\mathbb{P}})^q] = c^{q-1} = e^{c_H (q-1)}.$$

Moreover, for the case where $\lambda_t = \lambda(t)$, i.e λ is only a deterministic function of time, $\eta \equiv \xi \equiv 0$ is the solution to equation (3.3), and immediately one derives that

$$c_H = \frac{q}{2} \int_0^T \lambda^2(t)dt$$

which is also obtained by equation (3.5) and agrees with the findings in Hobson (2004).

Remark 3.5 Similarly, let us suppose that equation (3.4) holds and moreover, B and W are independent, $\lambda_t \equiv \lambda(Y_t, t)$, i.e. $\mu(S_t, Y_t, t) = \hat{\mu}(Y_t, t)S_t$ and $\sigma(S_t, Y_t, t) = \hat{\sigma}(Y_t, t)S_t$, and the "mean-variance trade-off process" $K_t := \int_0^t \lambda_t^2 dt$ is uniformly bounded, then one obtains the same result as in the example appearing in pages 1032–1036 in Grandits & Rheinländer (2002). It is an immediate consequence of Proposition 3.1.

In order to highlight the importance of Proposition 3.1 and prove the above claim, observe that for $\eta \equiv 0$ conditions *(i)* and *(ii)* are immediately satisfied and equation (3.3) is reduced to

$$\mathcal{E}(M^Y)_T = c \exp(-\frac{q}{2} \int_0^T \lambda^2(Y_t, t)dt).$$

Then, the Martingale Representation Theorem guarantees that there exists a solution. As a result, all conditions of Proposition 3.1 are satisfied and

$$V_T^{\text{opt}} = c(\mathcal{E}(-(q-1)\bar{\lambda} \cdot S)_T)^{p-1}$$

is the q-optimal measure. Moreover, V_T^{opt} can be rewritten as

$$V_T^{\text{opt}} = c \exp\left(-\frac{1}{2}\left(1 + \frac{1}{p-1}\right)\int_0^T \frac{\mu_t^2}{\sigma_t^2} dt\right) \mathcal{E}\left(-\frac{\mu}{\sigma} \cdot W\right)_T$$

which is the same as the representation given in Grandits & Rheinländer (2002), page 1034. Futhermore, one can show

$$\begin{aligned} c_H &= -\ln(\mathbb{E}[(\mathcal{E}((q-1)(\bar{\eta} - \bar{\lambda}) \cdot S)_T)^p]) = -\ln(\mathbb{E}[(\mathcal{E}((q-1)(\bar{\eta} - \bar{\lambda}) \cdot S)_T)^{p-1}]) \\ &= -\ln(\mathbb{E}[\exp(-\frac{q}{2} K_T)]) \end{aligned}$$

which agrees with the findings in Hobson (2004).

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